Mystery of Negations

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CS6840 presentation by
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Negations

Lower Bounds and Negations

No non-linear lower bounds are known for circuits using *NOT* gates and the effect of such gates on a circuit size remains to a large extent a mystery.

Minimum Negations

What is the minimum number of NOT gates required in a circuit computing f?

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What is the minimum number of *NOT* gates required in a circuit computing f?

$\Rightarrow \lceil \log(n+1) \rceil$ *NEGATIONS*

Theorem [Markov 1957]

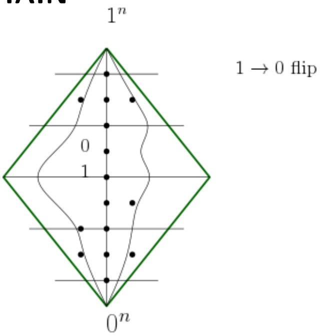
For every function f , the minimum number of NOT gates contained in a circuit computing f is precisely
$M(f) := \lceil log(d(f)+1) \rceil$

PRELIMINARIES

MONOTONOCITY

- $x, y \in \{0,1\}^n$ we say $x \leq y$ if $\forall i \; x_i \leq y_i$
- A function f is monotone if $x \leq y$ implies $f(x) \leq f(y)$

CHAIN



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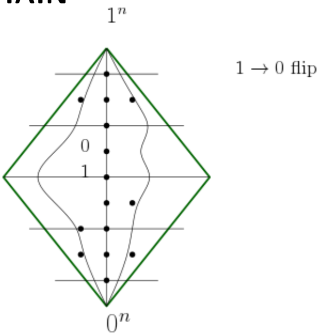

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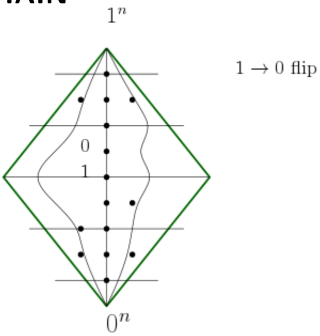

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- *Decrease* $d_Y(f)$ on a chain Y is no of indices i s.t. $f(y^i) > f(y^{i+1})$.
- *Decrease* $d(f)$ of f is maximum of $d_Y(f)$ over all chains Y

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Lower Bound

Let $neg(f)$ be the number of negation gates in the circuit computing f. We will first show that $neg(f) \geq \lceil \log(d(f)+1) \rceil$

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- Let g be fn computed on output of first negation
- $d_Y(g) \leq 1$

Proof

- If $d_Y(g) = 0$ the $g \equiv 0$ or $g \equiv 1$. Replace by constant 0 or 1.
- Otherwise, $\exists$ an i_0 s.t. $g(y^i) = 1$ for all $i \in I_1 = \{1, \ldots i_0\}$ and $g(y_i) = 0$ for all $i \in I_0 = \{i_0 + 1, \ldots, k\}$
- Depending $|I_1 \cap I(f)| \geq |I(f)|/2$ or not, replace the gate g by constant 0 or 1
- In both cases , the new fn f_1 has one fewer *NOT* gate and $d_Y(f_1) \geq |I(f)|/2$
- Do this for $r \leq \lceil \log(|I(f)| + 1) \rceil - 1$ steps and we will have contradiction.

Proof

Upper Bound

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ and $neg(f)$ be the number of negation gates in the circuit computing f.

To show

$$neg(f) \leq \lceil \log (d(f) + 1) \rceil$$

Proof by Induction on

$$M(f) := \lceil \log (d(f) + 1) \rceil$$

Base Case

$M(f) = 0 \Rightarrow d(f) = 0,$ so f is monotone and $neg(f) = 0$

Induction Step

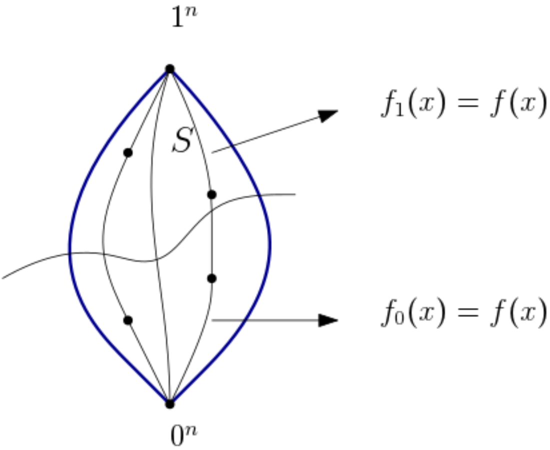
$neg(f') \leq M(f')$ for all boolean functions f' s.t. $M(f') \leq M(f) - 1$

Let S be set of all vectors $x \in \{0,1\}^n$ s.t. *for every chain* Y *starting with* x we have

$$d_Y(f) < 2^{M(f)-1} \tag{1}$$

We can also show that *every chain* Y *ending in a vector outside the set* S we have

$$d_Y(f) < 2^{M(f)-1} \tag{2}$$


1^n
S
$f_1(x) = f(x)$
$f_0(x) = f(x)$
0^n

Consider these 2 functions f_0 and f_1 as follows :-

$$f_1(x) = \begin{cases} f(x) & , if \ \ x \in S \\ 0 & , if \ \ x \notin S \end{cases}$$

and

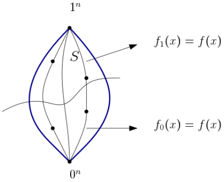
$$f_0(x) = \begin{cases} 1 & , if \ \ x \in S \\ f(x) & , if \ \ x \notin S \end{cases}$$

From (1) *and* (2) we have ,

$$d(f_1) \leq 2^{M(f)-1} - 1$$

$$d(f_0) \leq 2^{M(f)-1} - 1$$

f_1 : Upper part , all $f(x)$ and below the line all $0's$
f_0 : Lower part , all $f(x)$ and upper part has all $1's$



Hence by Induction Hypothesis,

$$M(f_1) = \lceil \log \left(d(f_1) + 1 \right) \rceil \leq M(f) - 1$$

$$M(f_0) = \lceil \log \left(d(f_0) + 1 \right) \rceil \leq M(f) - 1$$

Therefore remains to show

$$neg(f) \leq 1 + max\{neg(f_0), neg(f_1)\} \leq M(f)$$

Connector Function

Let $\mu(y, y', x)$ be a boolean function in $n+2$ variables $y, y', x_1 \dots x_n$. We say μ is a connector of two boolean functions $f_0(x)$ and $f_1(x)$ if for $i = 0, 1$

$$\mu(i, \neg i, x) = f_i(x)$$

that is, $\mu(0, 1, x) = f_0(x)$ and $\mu(1, 0, x) = f_1(x)$

Claim

Every pair of functions $f_0(x)$ and $f_1(x)$ has a connector μ such that $neg(\mu) \leq max\{neg(f_0), neg(f_1)\}$

Proof: Assume for now . . .

Let $s(x)$ be the characteristic function of S i.e.

$$s(x) = \begin{cases} 1 & , x \in S \\ 0 & , x \notin S \end{cases}$$

Note: $s(x)$ is monotone !!

Let μ be a connector of f_0 and f_1. Then

$$f(x) = \mu(s(x), \neg s(x), x)$$

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$$neg(f) \leq 1 + neg(\mu) = 1 + max\{neg(f_0), neg(f_1)\}$$
$$\Rightarrow neg(f) \leq M(f)$$

□

Why ??

$$x \in S \Rightarrow s(x) = 1$$

$$\Rightarrow \mu(1, 0, x) = f_1(x) = f(x)$$

and

$$x \notin S \Rightarrow s(x) = 0$$

$$\Rightarrow \mu(0, 1, x) = f_0(x) = f(x)$$

Hence ,

$$f(x) = \mu(s(x), \neg s(x), x)$$

Proof of Claim

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- Base Case $r = 0 \Rightarrow f_i$ are monotone and hence
  $\mu(y, y', x) = (y \wedge f_1) \vee (y' \wedge f_0)$ **[0 negations !!!]**

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- Base Case $r = 0 \Rightarrow f_i$ are monotone and hence
  $\mu(y, y', x) = (y \wedge f_1) \vee (y' \wedge f_0)$ **[0 negations !!!]**
- Induction Step
  $C_i(x)$ be the circuit with $neg(f_i)$ negations and computing $f_i(x)$
  - Replace first NOT gate in C_i by a var z, obtaining new circuit $C_i'(z, x)$ on $n + 1$ variables and computing $f_i'(z, x)$
  - $C_i'(z, x)$ has one NOT gate fewer
  - $neg(f_i') \leq r - 1$

Proof Contd

- Define $h_i(x)$ as monotone function computed before the first *NOT* gate. We have

$$f_i(x) = f_i'(\neg h_i(x), x)$$

- By Induction Hypothesis , $\exists$ connector boolean function $\mu'(y, y', z, x)$ (connector for pair f_0', f_1') s.t. $neg(\mu') \leq max\{neg(f_0'), neg(f_1')\} \leq r - 1$
- Replace var z with the function $Z(y, y', x) = \neg((y \wedge h_1(x)) \vee (y' \wedge h_0(x)))$ to obtain a new connector boolean function $\mu(y, y', x)$
  - $Z(0, 1, x) = \neg h_0(x)$ and $Z(1, 0, x) = \neg h_1(x)$
- $\mu(y, y', x)$ is a connector for f_0, f_1

Proof Contd

- Note that h_0 and h_1 are monotone
- $neg(\mu) \leq 1 + neg(\mu') \leq r$ [As Required]

  Remember $r = max\{neg(f_0), neg(f_1)\}$

Hence $neg(\mu) \leq max\{neg(f_0), neg(f_1)\}$ □

Theorem[Fischer's 1974]

If a function on n variables can be computed by a circuit of size of t, then it can be computed by a circuit of size at most $2t + \mathcal{O}(n^2 \log^2 n)$ using atmost $M(n) := \lceil \log(n+1) \rceil$ *NOT* gates

Proof:

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Proof:

- Push all the negations to the inputs (with care !!) [Size= 2t]
- $NEG(x_1, x_2, \ldots x_n) = (\neg x_1, \ldots \neg x_n)$ using just $M(n)$ negations and size $\mathcal{O}(n^2 \log^2 n)$
- $$\neg x_i = \bigwedge_{k=0}^{n} (\neg T_k^n(x) \vee T_{k,i}^n(x))$$

Proof Contd

- T_k^n is Threshold function and $T_{k,i}^n(x_1, \dots x_n) := T_k^{n-1}(x_1, \dots x_{i-1}, x_{i+1}, \dots x_n)$
- Remains to compute $\neg T(x) := (\neg T_1^n(x), \neg T_2^n(x), \dots \neg T_n^n(x))$ using atmost $\lceil \log(n+1) \rceil$ negations
- Hint:- $T(x) := (T_1^n(x), T_2^n(x), \dots T_n^n(x))$ can be computed by monotone circuits

Rest left as an exercise!

Motivation from Markov's

To what extend can we decrease the number of *NOT* gates in a circuit without a substantial increase in its size?

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To what extend can we decrease the number of *NOT* gates in a circuit without a substantial increase in its size?

Suppose a function f in n variables can be computed by a circuit of size polynomial in n , but for every circuit with $M(f)$ negations computing f requires superpolynomial size $(n^{\log n})$. What is then minimal number $R(f)$ of negations sufficient to compute f in polynomial size?

$R(f)$: Minimum no of negations sufficient to compute f in polynomial size

Fischer's result only implies that

$$M(f) - - - - - - R(f) - - - - - - \lceil \log(n+1) \rceil$$

where, $M(f) = \lceil \log(d(f)+1) \rceil$

Improvement to Fischer's Simulation

- Berkowitz and Valiant have shown that for *slice* functions, negations are almost useless i.e. can't lead to any superpolynomial savings

Will there be any superpolynomial savings at all using NOT gates?

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Razborov resolved this (long standing) problem.

1. There exist explicit monotone boolean function f s.t. $R(f) > 0$. The function is characteristic function of bipartite graphs containing a perfect matching.

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2. **Tardos Function** Non-monotone circuit (poly sized $m^{O(1)}$) and monotone circuit (size $2^{\Omega(m^{\frac{1}{8}})}$)

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   There exists monotone boolean function that can be computed with poly size , constant depth , unbounded fan in but can not be computed with monotone poly size constant depth circuits.

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Under additional restrictions following are the results by different authors.

1. Okolnishnikova(1982) and Ajtai and Gurevich (1987) There exists monotone boolean function that can be computed with poly size , constant depth , unbounded fan in but can not be computed with monotone poly size constant depth circuits.
2. Santha and Wilson(1993) In the class of constant-depth circuits, we need much more than $\lceil \log(n+1) \rceil$ negations . A multi output function that cannot be computed by constant depth using $o(\frac{n}{\log^{1+\epsilon} n})$

How many negations are enough to prove $P \neq NP$?

Markov-Fischer

To show that $P \neq NP$, it is enough to show a function $f : \{0,1\}^n \rightarrow \{0,1\}^n$ which is in NP and cannot be computed by any polynomial size circuit. By the results of Markov and Fischer it would be enough to prove a "weaker" result. Namely, let

$P^{(r)}$ = class of all functions $f : \{0,1\}^n \rightarrow \{0,1\}^n$ computable by poly-size circuits with atmost r NOT gates.

CLIQUE

Let *CLIQUE* be the monotone boolean function of $\binom{n}{2}$ variables which accepts a given input graph on n vertices iff it contains a clique on $n/2$ vertices . Since, $P \neq NP$ if $CLIQUE \notin P$, Markov-Fischer results imply that :

If $CLIQUE \notin P^{(r)}$ for $r = \lceil \log(n+1) \rceil$, then $P \neq NP$

Razborov's 1985

$$CLIQUE \notin P^{(r)} \text{ for } r = 0$$

Amano and Maruoka (2005) have shown a stronger result :

$$CLIQUE \notin P^{(r)} \text{ even for } r = \frac{1}{6} \log \log n$$

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- Unfortunately, Jukna(2004) showed that there are monotone functions $f \in P$ for which $R(f)$ is near to Markov's $\log n$ border.

Theorem [Jukna 2004]

There exists explicit feasible monotone functions $f_n : \{0,1\}^n \to \{0,1\}^n$ such that $R(f_n) \geq \log n - 9 \log\log n$

Thoughts and Questions

Thanks