On Alternation, VC-dimension and k-fold Union of Sets

Amit Roy[1] Jayalal Sarma[1]

EuroComb 2021

[1]Department of Computer Science and Engineering
Indian Institute of Technology Madras, Chennai, India

Vapnik-Chervonenkis Dimension - [VC71]

VAPNIK-CHERVONENKIS DIMENSION - [VC71]

- Let U be a universe and $\mathcal{F} \subseteq 2^U$.
- $\mathcal{F}$ is said to shatter a set $S \subseteq U$ if for all $S' \subseteq S$, $\exists F \in \mathcal{F}$, s.t. $S \cap F = S'$.

Vapnik-Chervonenkis Dimension - [VC71]

- Let U be a universe and $\mathcal{F} \subseteq 2^U$.
- $\mathcal{F}$ is said to shatter a set $S \subseteq U$ if for all $S' \subseteq S$, $\exists F \in \mathcal{F}$, s.t. $S \cap F = S'$.

The VC-dimension of $\mathcal{F}$ is said to be the largest d, such that $\mathcal{F}$ shatters a set S of size d.

Properties

Suppose, $\mathcal{F}$ shatters a set S. Then,

- $\text{VC}(\mathcal{F}) \geq |S|$.
- $|\mathcal{F}| \geq 2^{|S|}$
- $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \text{VC}(\mathcal{F}) \leq \text{VC}(\mathcal{G})$

Properties

Suppose, $\mathcal{F}$ shatters a set S. Then,

- $\text{VC}(\mathcal{F}) \geq |S|$.
- $|\mathcal{F}| \geq 2^{|S|}$
- $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \text{VC}(\mathcal{F}) \leq \text{VC}(\mathcal{G})$

Example:

- $\mathcal{F} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{c\}\}$
- $\text{VC}(\mathcal{F}) = 2$
- $S = \{b, c\}$
- Projection of S on $\mathcal{F}$: $\{S \cap F \mid F \in \mathcal{F}\})$

$$\Pi_{\mathcal{F}}(S) = \{\phi, \{b\}, \{c\}, \{b, c\}, \{c\}\}$$

Family in Our Context

1. Universe $U = \{0,1\}^n$
2. Hypothesis Class - $\mathcal{F} \subseteq 2^U$
3. Each concept $F \in \mathcal{F}$ is a set of positive inputs for a Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$

Family in Our Context

1. Universe $U = \{0,1\}^n$
2. Hypothesis Class - $\mathcal{F} \subseteq 2^U$
3. Each concept $F \in \mathcal{F}$ is a set of positive inputs for a Boolean function $f : \{0,1\}^n \to \{0,1\}$
4. Let $\mathcal{F}$ be a family then,
   - $\text{VC}(\mathcal{F}) \geq \frac{\log(|\mathcal{F}|)}{n}$ (From Sauer-Shelah Lemma)
   - $\text{VC}(\mathcal{F}) \leq \log(|\mathcal{F}|)$
5. VC-dimension has connection to PAC-learnability. Any (ϵ, δ) learning algorithm must use $\frac{\text{VC}(\mathcal{F})}{\epsilon}$ samples.

Known Bounds

Function Family	Upper Bound	Lower Bound	Remarks
Monotone Functions	$\binom{n}{n/2}$	$\binom{n}{n/2}$	[DPR06]
Monomials	n	n	[NS96]
Monotone Monomials	n	n	[NS96]
DNFs with terms size k	$O(n^k)$	$\Omega(n^k)$	[EHKV89]
k-Decision Lists	$O(n^k)$	$\Omega(n^k)$	[Riv87]
Symmetric Functions	$n+1$	$n+1$	[EHKV89]

Table: Bounds on VC-dimension of Families of Functions

Problem

What is the VC-dimension of family of non-monotone functions?

Problem

What is the VC-dimension of family of non-monotone functions?

We will see now a measure of *non-monotonicity* called *Alternation*.

Alternation

Monotone Boolean function

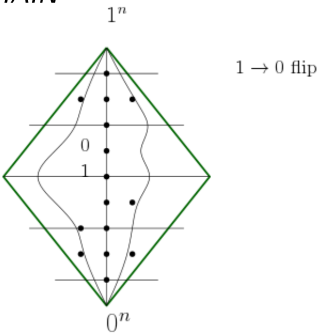
- $x, y \in \{0,1\}^n$ we say $x \preceq y$ iff $\forall i \; x_i \leq y_i$
- A function f is monotone if $x \preceq y$ implies $f(x) \leq f(y)$

Alternation

Monotone Boolean function

- $x, y \in \{0,1\}^n$ we say $x \preceq y$ iff $\forall i \; x_i \leq y_i$
- A function f is monotone if $x \preceq y$ implies $f(x) \leq f(y)$

CHAIN



- A Chain is an increasing sequence $y^1 \prec y^2 \ldots \prec y^k$ in the boolean hypercube.

Alternation of a Boolean function

1. Let $\mathcal{B}_n$ denote the n bit Boolean hypercube.
2. Consider a *chain* in $\mathcal{B}_n$, $y^0 \prec y^1 \prec \cdots \prec y^n$.
3. For a Boolean function f, alternation of f over a chain C of B_n denoted $\mathrm{alt}(f, C)$ is defined as
$\mathrm{alt}(f, C) = \left|\{i \mid f(y^{i-1}) \neq f(y^i), y^i \in C, i \in [n]\}\right|$

Alternation of a Boolean function

1. Let $\mathcal{B}_n$ denote the n bit Boolean hypercube.
2. Consider a *chain* in $\mathcal{B}_n$, $y^0 \prec y^1 \prec \cdots \prec y^n$.
3. For a Boolean function f, alternation of f over a chain C of B_n denoted $\text{alt}(f, C)$ is defined as $\text{alt}(f, C) = \left|\{i \mid f(y^{i-1}) \neq f(y^i), y^i \in C, i \in [n]\}\right|$

$$\boxed{\text{alt}(f) = \max_{C \in \mathcal{C}} \{\text{alt}(f, \mathcal{C})\}}$$

Alternation of a Boolean function

1. Let $\mathcal{B}_n$ denote the n bit Boolean hypercube.
2. Consider a *chain* in $\mathcal{B}_n$, $y^0 \prec y^1 \prec \cdots \prec y^n$.
3. For a Boolean function f, alternation of f over a chain C of B_n denoted $\text{alt}(f, C)$ is defined as $\text{alt}(f, C) = \left|\{i \mid f(y^{i-1}) \neq f(y^i), y^i \in C, i \in [n]\}\right|$

$$\boxed{\text{alt}(f) = \max_{C \in \mathcal{C}} \{\text{alt}(f, \mathcal{C})\}}$$

Note: Alternation of a non-constant Monotone Boolean function is 1.

Our Result #1: Family of alternation 1

Define a family,

$$\mathcal{F}_1 = \{f \mid \mathsf{alt}(f) \leq 1\}$$

$\mathcal{F}_1$ has only two kinds of Boolean functions.

Our Result #1: Family of alternation 1

Define a family,

$$\mathcal{F}_1 = \{f \mid \text{alt}(f) \leq 1\}$$

$\mathcal{F}_1$ has only two kinds of Boolean functions.

1. $f : \{0,1\}^n \to \{0,1\}$ is a *monotone* Boolean function

Our Result #1: Family of alternation 1

Define a family,

$$\mathcal{F}_1 = \{f \mid \text{alt}(f) \leq 1\}$$

$\mathcal{F}_1$ has only two kinds of Boolean functions.

1. $f : \{0,1\}^n \rightarrow \{0,1\}$ is a *monotone* Boolean function
2. Negation of $f : \{0,1\}^n \rightarrow \{0,1\}$ is a *monotone* Boolean function

Exact VC-dimension bound of $\mathcal{F}_1$

Theorem

Let $\mathcal{F}_1 = \{f \mid \text{alt}(f) \leq 1\}$. Then, $\text{VC}(\mathcal{F}_1) = \binom{n}{\lfloor n/2 \rfloor} + 1$

Exact VC-dimension bound of $\mathcal{F}_1$

Theorem

Let $\mathcal{F}_1 = \{f \mid \text{alt}(f) \leq 1\}$. Then, $\text{VC}(\mathcal{F}_1) = \binom{n}{\lfloor n/2 \rfloor} + 1$

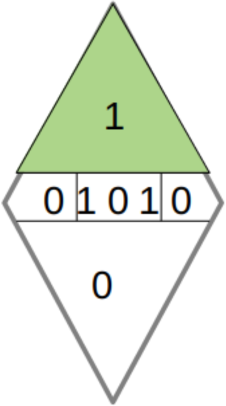
Proof :

Largest Shattered Set,

$$S = \{x \in \{0,1\}^n \mid wt(x) = \lfloor n/2 \rfloor\} \cup \{w\}$$

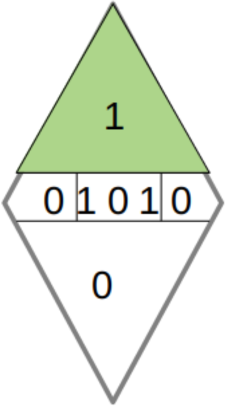
such that $wt(w) < n/2$.

Lower Bound



- We need to obtain all the $S' \subseteq S$ using only *monotone* or *negation of monotone* functions.

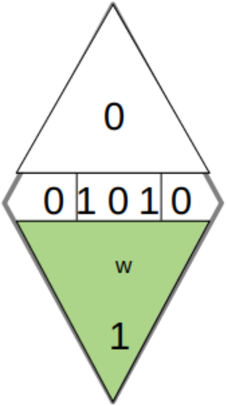
Lower Bound



- We need to obtain all the $S' \subseteq S$ using only *monotone* or *negation of monotone* functions.
- If $S' \subseteq S$ is an *antichain*. Then we obtain the set using a monotone function $f_{S'}$.

$$f_{S'} = \bigvee_{z \in S'} \bigwedge_{z_i = 1} x_i$$

Proof Contd.



- If $S' \subseteq S$ and $w \in S'$. Then we obtain the set using negation of a monotone function $f_{S'}$ as shown.

$$f_{S'} = \bigwedge_{z \in S'} \bigvee_{z_i = 1} \overline{x_i}$$

Upper Bound

Lemma

Any set S, $|S| \geq \binom{n}{n/2} + 2$ will have either of the two properties.

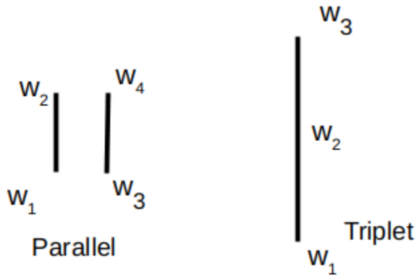
- Parallel Chain
- Triplet Chain

Upper Bound

Lemma

Any set S, $|S| \geq \binom{n}{n/2} + 2$ will have either of the two properties.

- Parallel Chain
- Triplet Chain



Upper Bound

Lemma

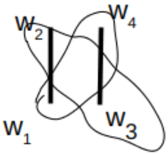
$\mathcal{F}_1$ cannot shatter a set if it has a *Parallel* or a *Triplet Chain*.

Upper Bound

Lemma

$\mathcal{F}_1$ cannot shatter a set if it has a *Parallel* or a *Triplet Chain*.

Proof:



Parallel

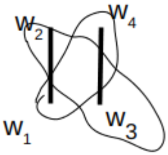
- No $f \in \mathcal{F}_1$ can obtain the subset $\{w_2, w_3\}$ and $\{w_1, w_4\}$

Upper Bound

Lemma

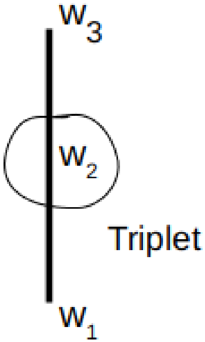
$\mathcal{F}_1$ cannot shatter a set if it has a *Parallel* or a *Triplet Chain*.

Proof:

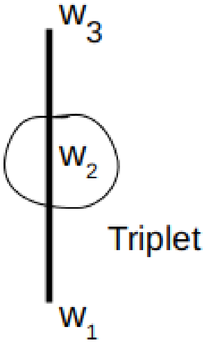


Parallel

- No $f \in \mathcal{F}_1$ can obtain the subset $\{w_2, w_3\}$ and $\{w_1, w_4\}$
- Any $f \in \mathcal{F}_1$ will also get $\{w_2, w_3, w_4\}$ or $\{w_1, w_2, w_3\}$.



- No $f \in \mathcal{F}_1$ will be able to obtain the subset $\{w_2\}$ and $\{w_1, w_3\}$.



- No $f \in \mathcal{F}_1$ will be able to obtain the subset $\{w_2\}$ and $\{w_1, w_3\}$.

- Hence largest shattered set is $\binom{n}{n/2} + 1$.

Our Result #2: VC-dimension of $\mathcal{F}_k$

Problem

Consider a family, $\mathcal{F}_k = \{f : \{0,1\}^n \to \{0,1\} \mid \mathrm{alt}(f) \leq k\}$*. What is the* VC-*dimension of the family* $\mathcal{F}_k$*?*

Lower Bound

Observe, $\mathcal{M} \subseteq \mathcal{F}_k \Rightarrow \text{VC}(\mathcal{F}_k) \geq \Omega(\binom{n}{n/2})$.

Theorem

Let $k > 1$. If $\mathcal{F}_k$ is the family of Boolean functions f such that $\text{alt}(f) \leq k$. Then, $\text{VC}(\mathcal{F}_k) \geq \sum_{i=n/2-k/2}^{n/2+k/2} \binom{n}{i}$.

Lower Bound

Observe, $\mathcal{M} \subseteq \mathcal{F}_k \Rightarrow \text{VC}(\mathcal{F}_k) \geq \Omega(\binom{n}{n/2})$.

Theorem

Let $k > 1$. If $\mathcal{F}_k$ is the family of Boolean functions f such that $\text{alt}(f) \leq k$. Then, $\text{VC}(\mathcal{F}_k) \geq \sum_{i=n/2-k/2}^{n/2+k/2} \binom{n}{i}$.

Proof :

- Shattered set is $S = \left\{x \in \{0,1\}^n \mid \frac{n}{2} - \frac{k}{2} \leq wt(x) \leq \frac{n}{2} + \frac{k}{2}\right\}$

Lower Bound

Observe, $\mathcal{M} \subseteq \mathcal{F}_k \Rightarrow \mathrm{VC}(\mathcal{F}_k) \geq \Omega(\binom{n}{n/2})$.

Theorem

Let $k > 1$. If $\mathcal{F}_k$ is the family of Boolean functions f such that $\mathrm{alt}(f) \leq k$. Then, $\mathrm{VC}(\mathcal{F}_k) \geq \sum_{i=n/2-k/2}^{n/2+k/2} \binom{n}{i}$.

Proof :

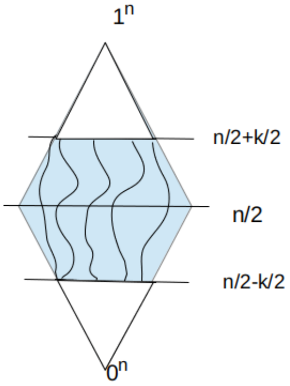
- Shattered set is $S = \left\{x \in \{0,1\}^n \mid \frac{n}{2} - \frac{k}{2} \leq wt(x) \leq \frac{n}{2} + \frac{k}{2}\right\}$
- $\forall S' \subseteq S, \exists F \in \mathcal{F}_k, (S \cap F = S')$

Lower Bound Proof

- Consider $S' \subseteq S$. Define ,

$$f_{S'}(x) = \begin{cases} 1 & , \ if \ \ x \in S' \\ 0 & , \ otherwise \end{cases}$$

- We claim, $\text{alt}(f_{S'}) \leq k$.
- Thus we obtain all $S' \subseteq S$



Alternation Characterization

Characterization of Alternation [BCO+15]

Let $f : \{0,1\}^n \to \{0,1\}$. Then there exists $k = \text{alt}(f)$ monotone functions $g_1, \ldots, g_k$ each from $\{0,1\}^n$ to $\{0,1\}$ such that

$$f(x) = \begin{cases} \oplus_{i=1}^{k} g_i & \text{if } f(0^n) = 0 \\ \neg \oplus_{i=1}^{k} g_i & \text{if } f(0^n) = 1 \end{cases}$$

First Upper Bound

Theorem [Son98, BEHW89, HW87]

Given k families of Boolean functions $\mathcal{F}_1, \mathcal{F}_2 \ldots, \mathcal{F}_k$, and a fixed Boolean function $f : \{0,1\}^k \to \{0,1\}$. Define,

$$\mathcal{F}(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k) = \{f(f_1(.), ..., f_k(.)) \mid f_i \in \mathcal{F}_i, i \in [k]\}$$

Let $d = \max_{i \in [k]}(\text{VC}(\mathcal{F}_i))$. Then,

$$\text{VC}(\mathcal{F}(\mathcal{F}_1, \ldots, \mathcal{F}_k)) \leq O(dk \log k)$$

First Upper Bound

Theorem [Son98, BEHW89, HW87]

Given k families of Boolean functions $\mathcal{F}_1, \mathcal{F}_2 \ldots, \mathcal{F}_k$, and a fixed Boolean function $f : \{0,1\}^k \to \{0,1\}$. Define,

$$\mathcal{F}(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k) = \{f(f_1(.), ..., f_k(.)) \mid f_i \in \mathcal{F}_i, i \in [k]\}$$

Let $d = \max_{i \in [k]}(\mathrm{VC}(\mathcal{F}_i))$. Then,

$$\mathrm{VC}(\mathcal{F}(\mathcal{F}_1, \ldots, \mathcal{F}_k)) \leq O(dk \log k)$$

- Using the characterization and this Theorem we obtain $\mathrm{VC}(\mathcal{F}_k) \leq O(k\binom{n}{n/2} \log k)$.

Improved Upper Bound

Theorem

Let $k > 1$. If $\mathcal{F}_k$ is the family of Boolean functions f such that $\text{alt}(f) \leq k$. Then, $\text{VC}(\mathcal{F}_k) \leq O\left(k\binom{n}{n/2}\right)$

Proof(Upper Bound)

Consider the family,

$$\mathcal{G} = \left\{ f \oplus g \mid f = \oplus_{i=1}^{k} f_i, f_i \in \mathcal{M}, g = const \right\}$$

where

$$g(x) = \begin{cases} 1 \ , & if \ \ f(0^n) = 1 \\ 0 \ , & if \ \ f(0^n) = 0 \end{cases}$$

$\mathcal{M}$ is Monotone family.

Proof(Upper Bound)

Consider the family,

$$\mathcal{G} = \left\{ f \oplus g \mid f = \oplus_{i=1}^{k} f_i, f_i \in \mathcal{M}, g = const \right\}$$

where

$$g(x) = \begin{cases} 1 \ , & if \ \ f(0^n) = 1 \\ 0 \ , & if \ \ f(0^n) = 0 \end{cases}$$

$\mathcal{M}$ is Monotone family.

Observation

Family $\mathcal{F}_k \subseteq \mathcal{G}$ and hence $\mathrm{VC}(\mathcal{F}_k) \leq \mathrm{VC}(\mathcal{G})$.

Bounding VC-dimension $\mathcal{G}$

1. $|\mathcal{G}| \leq |\mathcal{M}|^{k+1}$

Bounding VC-dimension $\mathcal{G}$

1. $|\mathcal{G}| \leq |\mathcal{M}|^{k+1}$
2. $\mathrm{VC}(\mathcal{G}) \leq log(|\mathcal{G}|) = (k+1)\log(|\mathcal{M}|)$
3. $|\mathcal{M}|$- Dedekind's Number !!
4. Due to *Kleitman et al.* [KM75]) :

$$\log(|\mathcal{M}|) \leq \binom{n}{n/2}\left(1 + O(\frac{\log n}{n})\right)$$

Bounding VC-dimension $\mathcal{G}$

1. $|\mathcal{G}| \leq |\mathcal{M}|^{k+1}$
2. $\text{VC}(\mathcal{G}) \leq log(|\mathcal{G}|) = (k+1)\log(|\mathcal{M}|)$
3. $|\mathcal{M}|$- Dedekind's Number !!
4. Due to *Kleitman et al.* [KM75]) :

$$\log(|\mathcal{M}|) \leq \binom{n}{n/2}\left(1 + O(\frac{\log n}{n})\right)$$

5. We get $\text{VC}(\mathcal{F}_k) \leq O(k\binom{n}{n/2})$

□

$\mathrm{VC}(\mathcal{F}_k) = \Theta(k\binom{n}{n/2})$

For $k = \Theta(\sqrt{n})$, $\binom{n}{n/2 \pm k} = \Theta(\binom{n}{n/2})$. Using this we have,

$\mathrm{VC}(\mathcal{F}_k) = \Theta(k\binom{n}{n/2})$

For $k = \Theta(\sqrt{n})$, $\binom{n}{n/2 \pm k} = \Theta(\binom{n}{n/2})$. Using this we have,

Corollary

Let $\mathcal{F}_k = \{f \mid \mathrm{alt}(f) \leq k\}$. For $k \leq \Theta(\sqrt{n})$,
$\mathrm{VC}(\mathcal{F}_k) = \Theta\left(k\binom{n}{n/2}\right)$.

Application to k-fold union

Problem

Consider a family $\mathcal{F}$. Define $\mathcal{F}^{k\cup} = \left\{\bigcup_{i=1}^{k} A_i \mid A_i \in \mathcal{F}\right\}$. How large can the VC-dimension of this family be?

Application to k-fold union

Problem

Consider a family $\mathcal{F}$. Define $\mathcal{F}^{k\cup} = \left\{\bigcup_{i=1}^{k} A_i \mid A_i \in \mathcal{F}\right\}$. How large can the VC-dimension of this family be?

- Blumer *et al* [BEHW89] and Haussler and Welzl [HW87] showed an upper bound of $O(dk \log k)$.

Application to k-fold union

Problem

Consider a family $\mathcal{F}$. Define $\mathcal{F}^{k\cup} = \left\{\bigcup_{i=1}^{k} A_i \mid A_i \in \mathcal{F}\right\}$. How large can the VC-dimension of this family be?

- Blumer *et al* [BEHW89] and Haussler and Welzl [HW87] showed an upper bound of $O(dk \log k)$.
- Eisentat and Angluin [EA07] show existence of a geometric family with VC-dimension at most d and the k-fold union has VC-dimension at least $\Omega(dk \log k)$.

Our Result #3: Application to k-fold union

We give a construction of k-fold union family which is

Our Result #3: Application to k-fold union

We give a construction of k-fold union family which is

- Explicit and non-geometric

Our Result #3: Application to k-fold union

We give a construction of k-fold union family which is

- Explicit and non-geometric
- VC-dimension at least $\Omega(dk)$

Our Result #3: Application to k-fold union

We give a construction of k-fold union family which is

- Explicit and non-geometric
- VC-dimension at least $\Omega(dk)$
- holds even when restricted to k-fold disjoint union

Construction of k-fold union

We now construct the family for k-fold union bound.

Construction of k-fold union

We now construct the family for k-fold union bound.

Lemma

Let $\mathcal{F}_{2k} = \{f : \{0,1\}^n \rightarrow \{0,1\} \mid \text{alt}(f) \leq 2k\}$. Then this family is same as $\mathcal{G} = \left\{\vee_{i=1}^{k} g_i \mid g_i : \{0,1\}^n \rightarrow \{0,1\}, \text{alt}(g_i) \leq 2\right\}$

Construction of k-fold union

We now construct the family for k-fold union bound.

Lemma

Let $\mathcal{F}_{2k} = \{f : \{0,1\}^n \rightarrow \{0,1\} \mid \text{alt}(f) \leq 2k\}$. Then this family is same as $\mathcal{G} = \left\{\vee_{i=1}^{k} g_i \mid g_i : \{0,1\}^n \rightarrow \{0,1\}, \text{alt}(g_i) \leq 2\right\}$

If we prove this lemma and show $\text{VC}(\mathcal{G}) = \Theta(k\binom{n}{n/2})$. We are done !!

Proof: $\mathcal{F}_{2k} \subseteq \mathcal{G}$

$$\text{alt}(f) \leq 2k \Rightarrow f = \bigoplus_{i=1}^{2k} f_i \quad (f_i \in \mathcal{M})$$

$$= \bigoplus_{i=1}^{k} (\neg f_{2i-1} \wedge f_{2i}) \vee (f_{2i-1} \wedge \neg f_{2i})$$

$$= \bigvee_{i=1}^{k} (\neg f_{2i-1} \wedge f_{2i}) \qquad (\textit{using} \quad f_i \rightarrow f_{i+1})$$

$$= \bigvee_{i=1}^{k} g_i \quad \text{such that } \text{alt}(g_i) \leq 2.$$

$\mathcal{G} = \mathcal{F}_{2k}$

Lemma

Let $g_1 : \{0,1\}^n \rightarrow \{0,1\}$ and $g_2 : \{0,1\}^n \rightarrow \{0,1\}$ with $\text{alt}(g_1) = k_1$ and $\text{alt}(g_2) = k_2$. Then $\text{alt}(g_1 \vee g_2) \leq k_1 + k_2$.

Shattering-Extremal Family

Let $\mathcal{F}$ be family and $Sh(\mathcal{F})$ be the set of shattered sets by $\mathcal{F}$.

Shattering-Extremal Family

Let $\mathcal{F}$ be family and $Sh(\mathcal{F})$ be the set of shattered sets by $\mathcal{F}$.

Lemma [Sau72]

For any family $\mathcal{F}$, $|Sh(\mathcal{F})| \geq |\mathcal{F}|$.

Shattering-Extremal Family

Let $\mathcal{F}$ be family and $Sh(\mathcal{F})$ be the set of shattered sets by $\mathcal{F}$.

Lemma [Sau72]

For any family $\mathcal{F}$, $|Sh(\mathcal{F})| \geq |\mathcal{F}|$.

A family is *Shattering Extremal* iff $|Sh(\mathcal{F})| = |\mathcal{F}|$.

Our Result #4: Extremal properties of Monotone family

Proposition

Let $\mathcal{M}$ be the family of monotone Boolean functions. Then $|Sh(\mathcal{M})| = |\mathcal{M}|$

Our Result #4: Extremal properties of Monotone family

Proposition

Let $\mathcal{M}$ be the family of monotone Boolean functions. Then $|Sh(\mathcal{M})| = |\mathcal{M}|$

Proof:

- *Maximal Antichain* $\leftrightarrow$ *Monotone function*
  - For each antichain, there is a unique Monotone function.
  - For each Monotone function , $\exists$ a unique maximal *antichain*.

Our Result #4: Extremal properties of Monotone family

Proposition

Let $\mathcal{M}$ be the family of monotone Boolean functions. Then $|Sh(\mathcal{M})| = |\mathcal{M}|$

Proof:

- *Maximal Antichain* $\leftrightarrow$ *Monotone function*
  - For each antichain, there is a unique Monotone function.
  - For each Monotone function , $\exists$ a unique maximal *antichain*.
- $\mathcal{M}$ can shatter any antichain set.

Our Result #4: Extremal properties of Monotone family

Proposition

Let $\mathcal{M}$ be the family of monotone Boolean functions. Then $|Sh(\mathcal{M})| = |\mathcal{M}|$

Proof:

- *Maximal Antichain* $\leftrightarrow$ *Monotone function*
  - For each antichain, there is a unique Monotone function.
  - For each Monotone function , $\exists$ a unique maximal *antichain*.
- $\mathcal{M}$ can shatter any antichain set.
- $|\mathcal{M}| = |Sh(\mathcal{M})|$

□

Properties of s-Extremal families

Properties of s-Extremal families

Problem (Meszaros-Ronyai Conjecture [MR13])

For an s-extremal *family $\mathcal{F}$, does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F} \setminus \{F\}$ is still* s-extremal*?*

Properties of s-Extremal families

Problem (Meszaros-Ronyai Conjecture [MR13])

For an s-extremal *family $\mathcal{F}$, does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F} \setminus \{F\}$ is still* s-extremal*?*

Conjecture holds for

1. Subset closed family
2. Shattering extremal families with VC-dimension 1

We show that Meszaros-Ronyai Conjecture holds for Monotone family.

Properties of s-Extremal families

Problem (Meszaros-Ronyai Conjecture [MR13])

For an s-extremal *family $\mathcal{F}$, does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F} \setminus \{F\}$ is still* s-extremal*?*

Conjecture holds for

1. Subset closed family
2. Shattering extremal families with VC-dimension 1

We show that Meszaros-Ronyai Conjecture holds for Monotone family.

- Remove the constant 1 function from family of Monotone functions $\mathcal{M}$.
- The family cannot shatter $\{0^n\}$ anymore. Why?

Properties of s-Extremal families

Problem (Meszaros-Ronyai Conjecture [MR13])

For an s-extremal *family $\mathcal{F}$, does there always exist a set $F \in \mathcal{F}$ such that $\mathcal{F} \setminus \{F\}$ is still* s-extremal*?*

Conjecture holds for

1. Subset closed family
2. Shattering extremal families with VC-dimension 1

We show that Meszaros-Ronyai Conjecture holds for Monotone family.

- Remove the constant 1 function from family of Monotone functions $\mathcal{M}$.
- The family cannot shatter $\{0^n\}$ anymore. Why?
- Thus $|Sh(\mathcal{M} \setminus g)| = |\mathcal{M}| - 1$.

Thoughts and Questions ?

Thanks

Eric Blais, Clément L. Canonne, Igor C. Oliveira, Rocco A. Servedio, and Li-Yang Tan.
Learning Circuits with few Negations.
In Naveen Garg, Klaus Jansen, Anup Rao, and José D. P. Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2015)*, volume 40 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 512–527, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

A. Blumer, A. Ehrenfeucht, D. Haussler, and Manfred K. Warmuth.
Learnability and the vapnik-chervonenkis dimension.
J. ACM, 36(4):929–965, October 1989.

Ariel D Procaccia and Jeffrey Rosenschein.

Exact vc-dimension of monotone formulas.
Neural Information Processing -Letters and Reviews, 10, 08 2006.

David Eisenstat and Dana Angluin.
The vc dimension of k-fold union.
Inf. Process. Lett., 101(5):181–184, March 2007.

Andrzej Ehrenfeucht, David Haussler, Michael Kearns, and Leslie Valiant.
A general lower bound on the number of examples needed for learning.
Information and Computation, 82(3):247 – 261, 1989.

David Haussler and Emo Welzl.
e-nets and simplex range queries.
Discrete & Computational Geometry, 2(2):127–151, 1987.

D. Kleitman and G. Markowsky.
On dedekind's problem: The number of isotone boolean functions. ii.
Transactions of the American Mathematical Society, 213:373–390, 1975.

Tamás Mészáros and Lajos Rónyai.
Shattering-extremal set systems of small vc-dimension.
International Scholarly Research Notices, 2013, 2013.

Thomas Natschläger and Michael Schmitt.
Exact vc-dimension of boolean monomials.
Information Processing Letters, 59(1):19 – 20, 1996.

Ronald L. Rivest.
Learning decision lists.
Machine Learning, 2(3):229–246, Nov 1987.

N Sauer.
On the density of families of sets.
Journal of Combinatorial Theory, Series A, 13(1):145–147, 1972.

E.D. Sontag.
Vc dimension of neural networks.
In C.M. Bishop, editor, *Neural Networks and Machine Learning*, pages 69–95. Springer, Berlin, 1998.

V. N. Vapnik and A. Ya. Chervonenkis.
On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities.
Theory of Probability & Its Applications, 16(2):264–280, 1971.